EMBEDDED CRACK IN A PLATE USING FINITE ELEMENT ALTERNATING TECHNIQUE

A Thesis Submitted
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by

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to the

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CERTIFICATE

This is to certify that the work entitled, "Embedded crack in a plate using Finite Element Alternating Technique" by N.J.Agrawal has been carried out under our supervision and has not been submitted elsewhere for a degree.

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February, 1989 (Nityanand Jagdishlal Agrawal)

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SYNOPSIS

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Initial cracks present in an engineering structure may lead to premature failure and evaluation of any new design of major structure from this point of view is drawing ever increasing attention of mechanical engineering designers. A variety of techniques are in use to predict fracture behaviour of engineering structures starting from simple closed form solutions to hybrid finite element methods and now Finite Element Alternating Technique (abbreviated as FEAT). Versatility , efficiency and ability to solve a variety of " Real World " problems are the major advantages of this technique. This technique is reported to be about ten times faster than Hybrid finite element method for the same accuracy . In the present work

this technique is applied to a three dimensional analysis of a finite thickness plate containing an embedded elliptical flaw. A three dimensional Finite Element Program is developed using twenty noded isoparametric elements. This together with Trefftz's solution for a three dimensional problem to a loaded elliptical crack forms the basis of finite element alternating technique. Results obtained have been compared with available numerical solutions, and an excellent agreement is observed for the embedded flaw problem under consideration.

CHAPTER I

INTRODUCTION

1.1 REVIEW OF PREVIOUS WORK

Design of large engineering structures such as submarine shells, large pressure vessels etc. involves a variety of design analyses to ensure reliable performance . An important part of this design analysis is fracture analysis to ensure performance in presence of initial manufacturing flaws of known sizes and flaws developed during service. This fracture analysis is primarily aimed at determining, acceptable loads for a given flaw size and shape and the also fatigue life as a function of flaw size and load.

This is often achieved by solving the problem under static loads only. Problem of crack propagation path and crack growth rate do not gain considerable importance from design analysis point of view as onset of crack growth is considered as the end of useful life of the component under design. It is thus often sufficient to find out the loads at which crack starts moving. Such static analysis is becoming a routine check in defense equipment design, several automotive component design, pressure vessel designs in nuclear and chemical plants, and aerospace structural designs.

It is therefore essential to have an efficient algorithm and software for carrying out fracture analysis for a variety of problems of practical importance. It is noted that practical situations often involve :

1.State of stress which could be analysed using a three dimensional technique.

2.surface or embedded crack configurations. Most of the applications mentioned in earlier paragraphs are likely to have mainly surface or embedded flaws resulting from fatigue ,

material defects , welding etc. Application and use of through crack is considerably restricted from the viewpoint of real life situations. It is also realized (1,19,20) that actual crack geometries are often modeled by an elliptical or semi- elliptical surfaces.

Thus keeping in view these practical stress states and crack configurations a three dimensional approach has become the necessity. With the criteria of practicality and efficiency as prime concern search for a suitable technique was made. Use of simple closed form solution is good for initial conceptual level analysis and for this purpose a virtual atlas of formulae (2) are available to solve a variety of problems. However, these often imply over simplifications and closed form solutions are nonexistent for most real life structures; for example, pipe junctions , forged drive train member etc. . Finite difference methods predates Finite element method and were used by Chen et al. (3), Baker (4), and Shmuely et al. (5) for stress intensity factors in a finite thickness central crack tension plate. Wilkins (6,7) used this technique for complex fracture problem such as an internally or surface flawed cylinder subjected to sudden pressurization. These methods are not recommended due to enormous and prohibitively large computer time required though still used some times.

There are three dimensional finite element methods using following approaches:

1. fine mesh of constant strain solid brick elements
2. Elements with $\sqrt{r}$ singularity , where r is the distance from the crack tip.
3. Collapsed quarter point elements.

Using densely packed three dimensional constant strain solid brick elements along the curved crack front by Miyamoto et al. (8) for proper modeling of the $1 / \sqrt{r}$ singularity results in an inefficient use of computer time (9) . Computational efficiency is improved by incorporating $1 / \sqrt{r}$ strain singularity

in displacement formulation of elements. Such elements were used for Mode I problem by Newman (10) and multimode surface flaw problem in pressurized cylinders by Raju (11) .These elements required addition of special subroutines to a general three dimensional FEM code. However, use of collapsed quarter point isoparametric elements (12,13) did not require any such additions and commercial packages like SAP IV (14) were used successfully to solve the embedded crack problem. The maximum errors in numerical results, when using collapsed quarter point elements, when compared with the theoretical results (15) were 5% and 7% for a circular crack and an elliptical crack with an aspect ratio of 1.5 respectively.

Displacement Hybrid Finite Element Method was used for three dimensional analysis of surface flaws in thick walled reactor pressure vessels and aerospace structural components by Atluri et al. (16,17) .This method is based on modified variational principle of the total potential energy such that exact asymptotic form of solution for singular stresses and strains in elements near the crack boundary is incorporated. The variational formulation leads to final algebraic equations which contain generalized global displacements and K stress intensity factors as unknown variable. Thus these equations upon solving, directly give K_I ,K_{II} ,K_{III} , along with general displacements u,v,w at the crack front .This hybrid displacement FEM is computationally more efficient than those using quarter point collapsed elements.

Finally, the Finite Element Alternating Technique (hence forth referred to as FEAT) which has been employed in present work, originally developed by Kobayashi et al.(18) requires two analytical solutions as follows:

(1) Solution to the problem of an infinite solid with an elliptical crack subjected to cubic normal loading on the crack surface. This limitation on the type of loading was due to the non-availability of a general solution for a general polynomial

type of normal loading on the crack face, and

(2) Solution to a semi infinite body subjected to uniform normal and shear stress over a rectangular portion of the surface.

In this work limitation of solution 1 was one of the reasons for causing about 10% error which was rather poor as compared with three dimensional Hybrid FEM (16) . In 1981 Vijay kumar and Atluri (19) derived a generalized solution procedure for an embedded elliptical crack subject to arbitrary crack-face tractions in an infinite solid. This procedure could be manually evaluated for a simple linear pressure variation of load on the crack face but required the use of a digital computer for a higher order pressure variation. And until 1983 when Nishioka et al. (20) evolved a generic procedure to numerically evaluate the required Elliptic Integrals in solution 1 , FEAT was considered as inaccurate though an efficient technique. In their approach, Nishioka replaced solution 1 and 2 of (18) by

(1) Solution to the elliptical crack subjected to arbitrary normal and shear loading on the crack surface , in an infinite solid, and

(2) Solution of uncracked body using a general numerical solution technique such as finite element method.

It was proved (19,20) that FEAT is not only a versatile technique in its application and capable of solving practical problems but also about 10 times faster than Hybrid Finite Element Method. Further this technique has been successfully used to solve multiple surface cracks problem in pressure vessels by O'Donoghue (1).

1.2 OBJECTIVE AND SCOPE OF PRESENT WORK

The objective of present work is to determine stress intensity factors along elliptical crack boundary in a finite plate containing an embedded crack using three dimensional Finite Element Alternating Technique. Tensile loads are considered on the plate such that the problem is restricted to mode I analysis.

In the present work loading on the crack face in solution 1 expressed as a fourth order polynomial. The work involves implementation of three dimensional Finite Element Method and solution 1 of FEAT in a single program . Twenty - noded isoparametric elements have been used in FEM analysis and multiple linear regression is carried out in order to determine stresses at the surface of elements.

Chapter II describes the FEAT technique as used in present work. Chapter III describes the program structure and related important features of program development for Finite Element Alternating Method .Also briefly explained are the variety of numerical techniques used in the program.

Finally, in Chapter IV results have been included and discussed. Chapter V presents conclusions drawn and suggestions for further work.

CHAPTER II

FINITE ELEMENT ALTERNATING TECHNIQUE

It is mentioned in chapter I that FEAT is an alternating use of FEM and a closed form solution for a loaded elliptical crack in an infinite body subjected to arbitrary crack face tractions. Therefore, before discussing FEAT in any further details the two solution procedures are described briefly.

2.1 FINITE ELEMENT METHOD

Since considerable amount of literature is available not only on the method (22) itself but also on programming the method (23), here only a brief treatment is included in order to highlight the salient features of the FEM as used in present work. The three dimensional Finite Element Method used has three displacement components as three degrees of freedom of each node. A twenty (20) noded isoparametric solid element is used in the present work. As FEM is being used for uncracked plate only there is no need of any singularity or special element in this method. This probably is the reason behind the high computational efficiency of FEAT. The displacement vector of an element is expressed in terms of the shape functions N_i and nodal displacement values as under

$$\{ U \} = \sum_{i=1}^{20} N_i \quad u_i$$

$$\{ V \} = \sum_{i=1}^{20} N_i \quad v_i \tag{2.1}$$

$$\{W\} = \sum_{i=1}^{20} N_i \quad w_i$$

where shape functions N_i are expressed in terms of local coordinates S,T and U as under (22)

for corner nodes with $S_i = +/-1$, $T_i = +/-1$, $U_i = +/-1$

$$N_i = (1+S_\emptyset)(1+T_\emptyset)(1+U_\emptyset)(S_\emptyset+T_\emptyset+U_\emptyset-2) \; / \; 8.\emptyset \qquad (2.2)$$

for midside nodes with $S_i = \emptyset$, $T_i = +/-1$, $U_i = +/-1$

$$N_i = (1-S^2)(1+T_\emptyset)(1+U_\emptyset) \; / \; 4.\emptyset \qquad (2.3)$$

for midside nodes with $T_i = \emptyset$, $S_i = +/-1$, $U_i = +/-1$

$$N_i = (1-T^2)(1+S_\emptyset)(1+U_\emptyset) \; / \; 4.\emptyset \qquad (2.4)$$

for midside nodes with $U_i = \emptyset$, $S_i = +/-1$, $T_i = +/-1$

$$N_i + (1-U^2)(1+S_\emptyset)(1+T_\emptyset) \; / \; 4.\emptyset \qquad (2.5)$$

$$\text{where } S_\emptyset = S \, . \, S_i$$

$$T_\emptyset = T \, . \, T_i \qquad (2.6)$$

$$U_\emptyset = U \, . \, U_i$$

For calculation of stresses at the Gauss points nodal displacements and strain matrices are generally used ; however, for calculation of stresses at the surface of the elements as would be needed in FEAT this approach leads to highly inaccurate stresses and is inadequate. Therefore multiple linear regression analysis (24) has been used to calculate stresses on surface of the element . The stress component of interest σ_{33} is expressed

as

$$\sigma_{33} = a_0 + a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 \qquad (2.7)$$

where x_1, x_2, x_3 are the global coordinates of the point at which stress is to be calculated. The Linear Regression Algorithm is used to find coefficients a_0 to a_3 of multiple linear regression for the desired elements.

It is important to mention here that Frontal solving technique for solving final simultaneous equations of FEM have been employed in the present work and the routine (23) used is suitable for only symmetric system of equations. Use of the algorithm by Nishioka, et al (20) for equation solving was developed by Mondkar and Powell (25) and is also a powerful out of core equation solver for FEM applications and is based on Crout's algorithm. Though no definite comparison between the two algorithms could be made, Frontal technique has been used in present work for following reasons

1.Global Stiffness matrix does not have to be assembled in Frontal technique as in case of Nishioka's approach there by requiring less of core memory.

2.FRONT as used here has re-solution facility to greatly reduce solution time for subsequent iterations, which is claimed to be the main advantage of algorithm used by Nishioka (20).

Program developed interpolates coordinates of all midside nodes if they lie on a straight edge there by avoiding entry of coordinates of all nodes manually. It is to be noted that even though the input of coordinates of each node is a semi-automatic one , developing a new mesh does call for manual input of all connectivity and fixity data.

2.2 SOLUTION PROCEDURE FOR AN ELLIPTICAL CRACK IN AN INFINITE BODY SUBJECTED TO ARBITRARY CRACK FACE TRACTIONS.

An generalized solution procedure for an elliptical crack in an infinite body subjected to arbitrary crack face tractions was developed by Vijaykumar and Atluri (19) . This solution is the second solution used in the finite element alternating technique. This solution is explained and relevant portions are derived in following paragraphs. The presentation of algebraic details of the solution procedure is however, kept to a minimum in the interest of clarity as well as reasons of space.

This solution procedure is based on TREFFTZ FORMULATION (14) . Let $u_x(x=1,2,3)$ and σ_{xy} $(x,y=1,2,3)$ denote displacements and stresses,respectively in a homogeneous, isotropic linear elastic solid. The stress-displacement relations are, by Hooke's law, in rectangular coordinates $X_\alpha(\alpha=1,2,3)$

$$\sigma_{\alpha\beta} = G\ (u_{\alpha,\beta} + u_{\alpha,\beta}\ \frac{2.v}{1-2.v}\ \delta_{\alpha\beta}\ u_{\Gamma,\Gamma}\ ) \qquad (2.8)$$

where G and v are the shear modulus and Poisson's ratio of the material , respectively, and $\delta_{\alpha\beta}$ is dirac delta function. The Navier displacement equations of equilibrium in the absence of body forces are

$$u_{\beta,\beta\alpha} + (1-2.v)\ u_{\alpha,\beta\beta} = 0 \qquad (2.9)$$

In the foregoing the notation,$(\ )_{,\alpha}$ and $(\ )_{,\alpha\beta}$ have been employed to denote first order and second order differentiation w.r.t. α and α,β respectively.

Let R be the region of discontinuity in the plane $X_3=0$ such that, after deformation, the material inside R breaks up

with free upper and lower surfaces, and remains continuous outside R. To deal with such a problem it is convenient to consider it's complementary problem in which the surface of region of discontinuity is subjected to arbitrary tractions $\sigma_{3\alpha}$.

It is well known that (26) the solution for the problem can be expressed in terms of harmonic functions θ and ϕ_α (α=1,2,3) in the form

$$u_\alpha = \phi_\alpha + x_3 \theta_{,\alpha} \tag{2.10}$$

so that the equations (2.9) are satisfied if

$$\phi_{\alpha,\alpha} + (3 - 4v)\theta_{,3} = 0 \tag{2.11}$$

The stress components in terms of ϕ_α and θ are

$$\sigma_{\alpha\beta} = G\left[\phi_{\alpha,\beta} + \phi_{\beta,\alpha} + \delta_{\alpha 3}\theta_{,\beta} + \delta_{\beta 3}\theta_{,\alpha} + 2x_3\theta_{,\alpha\beta} + \delta_{\alpha\beta}\frac{2v}{(1-2v)}(\phi_{k,k} + \theta_{,3})\right] \tag{2.12}$$

The problem is further simplified by expressing θ and ϕ in the form

$$\theta = \nabla . f = f_{\alpha,\alpha} \tag{2.13}$$

$$\phi_1 = (1 - 2v)(f_{1,3} + f_{3,1}) - (3 - 4v)f_{1,3} \tag{2.14}$$

$$\phi_2 = (1 - 2v)(f_{2,3} + f_{3,2}) - (3 - 4v)f_{2,3} \tag{2.15}$$

$$\phi_3 = -(1 - 2v)(f_{1,1} + f_{2,2}) - 2(1 - v)f_{3,3} \tag{2.16}$$

The stress components $\sigma_{\alpha\beta}$ in terms of f_α (α=1,2,3) are given by

$$\sigma_{11} = 2G[f_{3,11} + 2\nu f_{3,22} - 2f_{1,31} - 2\nu f_{2,32} + x_3(\nabla . f)_{,11}] \quad (2.17a)$$

$$\sigma_{22} = 2G[f_{3,22} + 2\nu f_{3,11} - 2f_{2,32} - 2\nu f_{1,31} + x_3(\nabla . f)_{,22}] \quad (2.17b)$$

$$\sigma_{12} = 2G[(1-2\nu)f_{3,12} - (1-\nu)(f_{1,32} + f_{2,13}) + x_3(\nabla . f)_{,12}] \quad (2.17c)$$

$$\sigma_{33} = 2G[- f_{3,33} + x_3(\nabla . f)_{,33}] \quad (2.17d)$$

$$\sigma_{31} = 2G[-(1- \nu)f_{1,33} +\nu(f_{1,11} + f_{2,21})+ x_3(\nabla . f)_{,13}] \quad (2.17e)$$

$$\sigma_{32} = 2G[-(1- \nu)f_{2,33} +\nu(f_{1,12} + f_{2,22})+ x_3(\nabla . f)_{,23}] \quad (2.17f)$$

where

$_{,1}$ $_{,2}$ and $_{,3}$ denote differentiation w.r.t. x_1, x_2 and x_3 respectively and

$$\nabla . f = f_{1,1} + f_{2,2} + f_{3,3} \quad (2.18)$$

Now the problem is reduced to finding appropriate potential functions $f_\alpha(\alpha = 1,2,3)$. Suppose that x_1 and x_2 are the cartesian coordinates in the plane of the elliptical crack and x_3 is normal to the crack-plane as shown in Fig. 2.1 such that :

$$(x_1/a_1)^2 + (x_2/a_2)^2 = 1, \ a_1 > a_2 \quad (2.19)$$

describes the border of the elliptical crack of aspect ratio (a_1/a_2). The ellipsoidal coordinates $\Phi_i(i=1,2,3)$ are more convenient in describing the above geometry and are defined as the roots of the cubic equation

$$w(s) = 1 - \frac{x_1^2}{a_1^2 + s} - \frac{x_2^2}{a_2^2 + s} - \frac{x_3^2}{s} = 0 \quad (2.20)$$

where Eq. (2.20) is written in an alternate form:

$$w(s)= P(s)/Q(s) \tag{2.21}$$

and

$$p(s)= (s - \Phi_1)(s - \Phi_2)(s - \Phi_3) \tag{2.22a}$$

$$Q(s)= s\ (s + a_1{}^2)(s + a_2{}^2) \tag{2.22b}$$

where

$$\infty > \Phi_3 \geq 0 \geq \Phi_2 \geq - a_2{}^2 \geq \Phi_1 \geq - a_1{}^2$$

the elliptic boundary (Fig.2.1) in the plane $x_3 = 0$ corresponds to the curve $\Phi_3 = \Phi_2 = 0$. The crack surface itself namely the region inside ellipse in the plane $x_3 = 0$ is now given in a simple manner by the surface $\Phi_3 = 0$ .The three potential functions $f_\alpha(\alpha=1,2,3)$ of the Trefftz's formulation are taken to be :

$$f_\alpha = \sum_k \sum_l C_{\alpha,k,l}\, F_{kl} \tag{2.23}$$

$$F_{kl} = \frac{\delta^{k+l}}{\delta x_1{}^k\ \delta x_2{}^l} \int_{\Phi_3}^{\infty} [w(s)]^{k+l+1} \frac{ds}{\sqrt{Q(s)}} \tag{2.24}$$

In Eq. (2.23) $C_{i,k,l}$ are undetermined coefficients and the commas are introduced for convenience only.

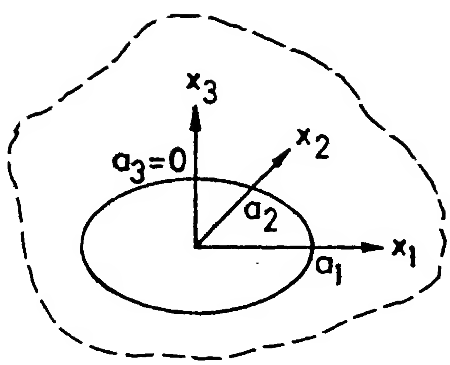

FIG. 2-1 ELLIPTICAL CRACK IN AN INFINITE SOLID

Denoting by $f_{\alpha,\beta}$ the partial derivatives of f_α with respect to $x_\beta (\beta = 1,2,3)$, the first ,second , and third partial derivatives of f_α are expressed as:

$$f_{\alpha,\beta} = \sum_k \sum_l C_{\alpha,k,l} F_{kl,\beta} \tag{2.25}$$

$$f_{\alpha,\beta\delta} = \sum_k \sum_l C_{\alpha,k,l} F_{kl,\beta\delta} \tag{2.26}$$

$$f_{\alpha,\beta\delta\Gamma} = \sum_k \sum_l C_{\alpha,k,l} F_{kl,\beta\delta\Gamma} \tag{2.27}$$

By successive differentiation , it can be shown from Eq. (2.24) that

$$F_{kl} = \int_{\Phi 3}^{\infty} \frac{\delta^{k+l} w^{k+l+1}}{\delta x_1^{k} \delta x_2^{l}} \frac{ds}{\sqrt{Q(s)}} \equiv \int_{\Phi 3}^{\infty} \delta_1^{k1} \delta_2^{l1} \delta_3^{m1}\, w^{k+l+1} \frac{ds}{\sqrt{Q(s)}} \tag{2.28}$$

wherein

$$k_1 = k;\ l_1 = l;\ m_1 = 0 \tag{2.29}$$

In Eq. (2.28) the additional notation that $\delta_\alpha{}^j$ implies the jth partial derivatives w.r.t. x_α has been used. Similarly , the first order partial derivatives of F_{kl} w.r.t. x_α $(\alpha =1,2,3)$ can be expressed by

$$F_{kl,\beta} = \int_{\Phi 3}^{\infty} \delta_1^{k1} \delta_2^{l1} \delta_3^{m1}\, w^{k+l+1} \frac{ds}{\sqrt{Q(s)}} \tag{2.30}$$

where

$$k_1 = k + \delta_{1\beta}\ ;\ l_1 = l + \delta_{2\beta}\ ;\ m_1 = \delta_{3\beta} \tag{2.31}$$

In case of second and third partial derivatives, following

could be derived:

$$F_{kl,\beta\Gamma} = \int_{J}^{\infty} \Phi 3\ \delta_1^{k1}\delta_2^{l1}\delta_3^{m1}\ w^{k+l+1}\ \frac{ds}{\sqrt{Q(s)}} \quad + \quad F^{\emptyset}_{kl\beta\Gamma} \quad (2.32)$$

where $k_1 = k + \delta_{1\beta} + \delta_{1\Gamma}$; $l_1 = l + \delta_{2\beta} + \delta_{2\Gamma}$; $m_1 = \delta_{3\beta}+\delta_{3\Gamma}$

$$F_{kl,\beta\Gamma\delta} = \int_{J}^{\infty} \Phi 3\ \delta_1^{k1}\delta_2^{l1}\delta_3^{m1}\ w^{k+l+1}\ \frac{ds}{\sqrt{Q(s)}} + \frac{\delta F^{\emptyset}_{kl\beta\Gamma}}{\delta x_\delta} + G_{\emptyset} \quad (2.33)$$

where $k_1 = k_\emptyset + \delta_{1\delta}$; $l_1 = l_\emptyset + \delta_2$; $m_1 = m_\emptyset + \delta_{3\delta}$.

where $k_\emptyset = k + \delta_{1\beta} + \delta_{1\Gamma}$; $l_\emptyset = l + \delta_{2\beta} + \delta_{2\Gamma}$; $m_\emptyset = \delta_{3\beta} + \delta_{3\Gamma}$.

Numerical evaluation of terms $G^{\emptyset}$, $F_{kl\beta\Gamma}$ and their derivatives is vital to determination of stresses in the body due to crack loading . These numerical procedures are covered in detail in (20). Also it should be noted that for evaluation of second and third order derivatives of function F_{kl} a generic integral of the following type needs to be frequently evaluated.

$$\int_{J}^{\infty} \Phi 3\ \delta_1^{k1}\ \delta_2^{l1}\ \delta_3^{m1}\ w^{k+l+1}\ \frac{ds}{\sqrt{Q(s)}} \qquad (2.34)$$

This can be achieved by expanding w^{k+l+1} in terms of $x^2{}_\alpha$ and carrying out term by term differentiations. Then one gets

$$\int_{\Phi_3}^{\infty} \delta^{k1}\,\delta^{l1}\,\delta^{m1}\,w^{k+l+1} \frac{ds}{\sqrt{Q(s)}} = (k+l+1)! \sum_{p=\emptyset}^{k+l+1} \sum_{q=\emptyset}^{p} \sum_{r=\emptyset}^{q} (-1)_p \cdot$$

$$\cdot \frac{(2p-2q)!(2q-2r)!(2r)!\; x^{2p-2q-k}\; x^{2q-2r-l}\; x^{2r-m}}{(k+l+1-p)!(p-q)!(q-r)!(r)!(2p-2q-k_1)!(2q-2r-l_1)!(2r-m_1)!}$$

$$\cdot J_{p-q,q-r,r}(\Phi_3) \qquad (2.35)$$

where (.) denotes multiplication and

$$J_{p-q,q-r,r}(\Phi_3) = \int_{\Phi_3}^{\infty} \frac{ds}{(s+a^2{}_1)^{p-q}\,(s+a^2{}_2)^{q-r}\,(s+a^2{}_3)^{r}\,\sqrt{Q(s)}} \qquad (2.36)$$

Thus one of the key algebraic steps in the successful application of FEAT is evaluation of generic elliptic integrals in equation (2.36). Nishioka et al (2Ø) have developed a systematic procedure for the evaluation of this elliptic integral.

2.3 EVALUATION OF GENERIC ELLIPTIC INTEGRALS

In general the integrals in Eqs. 2.36 could be expressed in a closed form using incomplete elliptic integrals of the first and second type and Jacobian elliptic functions, however, development of such expressions involves a lot of algebraic work and therefore there is a need for a systematic procedure for evaluation of these integrals as a function of parameter p, q, r. Thus rewriting equation (2.36) using Jacobian elliptic functions as

$$J_{p-q,q-r,r} \equiv \frac{2}{a^{2p+1}}\, L_{p,q-r,r} \qquad (2.37)$$

where

$$L_{p,q-r,r} = \frac{1}{(2r-1)k'^2}\{(sn^{2p+1}u)(nc^{2r-1}u)(nd^{2q-2r-1}u)\ u1$$

$$+ [2(-p+r-1) + 2(p-q-r+2)\ k^2]L_{p,p-q,r-1}$$

$$+ k^2(-2p+2q-3)\ L_{p,q-r,r-2}\}. \qquad (2.38)$$

$$k^2 = (a^2{}_1 - a^2{}_2)\ /\ a^2{}_1 \qquad (2.39a)$$

$$sn^2\ u_1 = a^2{}_1\ /\ (a^2{}_1 + \Phi_3) \qquad (2.39b)$$

sn(u),nc(u),nd(u) are Jacobian elliptic functions.

However, lowest order starting values of $L_{p,q-r,r-1}$ and $L_{p,q-r,r-2}$ are needed in order to evaluate $L_{p,q-r,r}$. The lowest order starting values are

$$L_{p,q,-1} = \int_{\emptyset}^{u1} (sn^{2p}u)\ (nd^{2q}u)\ (nc^{-2}u)\ du \qquad (2.4\emptyset)$$

$$L_{p,q,-2} = \int_{\emptyset}^{u1} (sn^{2p}u)\ (nd^{2q}u)\ (nc^{-4}u)\ du \qquad (2.41)$$

These integrals could be reduced (2Ø) to

$$L_{p,q,-1} = \frac{1}{k^{2p+2}} \sum_{j=\emptyset}^{p} \sum_{\Gamma=\emptyset}^{1} \frac{(-1)^{j+\Gamma+1}\ k'^{2\,2(1-\Gamma)}p!}{(p-j)!j!\ (1-\Gamma)!\Gamma!} I_{2(q-j-\Gamma)}$$

(2.42)

$$L_{p,q,-2} = \frac{1}{k^{2p+4}} \sum_{j=0}^{p} \sum_{\Gamma=0}^{2} \frac{(-1)^{j+\Gamma+2} k'^{2(2-\Gamma)} p!}{(p-j)!j!\ (2-\Gamma)!\Gamma!} I_{2(q-j-\Gamma)} \tag{2.43}$$

where

$$I_{2m} = \int_{0}^{u1} nd^{m}\, u \quad du \tag{2.44}$$

It is worth mentioning that in this series expansion of equation (2.42) denominator contains the term $(1-\Gamma)!$ and not $(2-\Gamma)!$ as given in (20). This has been verified by expanding integrals in Eq. (2.40 and 2.41) in terms of integrals I_{2m} (27) of Eq. (2.44).

Finally values of integral I_{2m} for various positive and negative values of m are to be determined. This can be done if we determine

1. Inverse of Jacobian elliptic function sn in equation (2.39b) to determine u_1.

2. Incomplete elliptic integral of second kind $E(u_1)$.

Numerical procedure to invert Jacobian elliptic function and determine incomplete elliptic integrals of first and second kind are discussed in section 2.6. Remaining details for determining I_{2m} could be found in (20,27). Thus in order to determine generic elliptic integral of Eq. (2.36) various arrays of integrals as outlined above need to be evaluated .

2.4 RELATION BETWEEN CRACK-FACE TRACTIONS AND POTENTIAL FUNCTIONS

The tractions along the crack surface is expressed in the following polynomial form

$$\sigma_{3\alpha} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} A^{(i,j)}_{\alpha,m-n,n}\, x^{2m-2n+i}\, x^{2n+j} \;;\quad (\alpha=1,2,3) \qquad (2.45)$$

so that the values i,j specify the symmetries of the load with respect to the axes of th ellipse. The solution in terms of potential function is assumed in the form

$$f_{\alpha} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{M} \sum_{l=0}^{k} C^{(i,j)}_{\alpha,k-l,l}\; F_{2k-2l+i,2l+j} \;;\quad (\alpha=1,2,3) \qquad (2.46)$$

The potential functions in Eq. (2.23) are to be understood as that k and l replace (2k-2l+i) and (2l+j) respectively. It can be shown (20) that following linear algebraic equations result for a Mode I problem, between A coefficients of Eq. (2.45) and C coefficients of Eq. (2.46).

$$\sum_{i=0}^{1} \sum_{j=0}^{1} \{A^{(i,j)}_{3,m-n,n}\} = \sum_{i=0}^{1} \sum_{j=0}^{1} \Big\{ 2G \frac{(-1)^{m+i+j+1}}{(2m-2n+i)!(2n+j)!} \sum_{k=m}^{M} \sum_{l=0}^{k} (-1)^{k} \qquad (2.47)$$

$$[I^{(i,j)} J_{L1+1,L2,\emptyset}(\emptyset) + I^{(i,j)} J_{L1,L2+1,\emptyset}(\emptyset)]\; C^{(i,j)}_{3,k-l,l} \Big\};$$

$$m = \emptyset,1,\ldots.,\; M \qquad (2.48)$$

$$n = \emptyset,1,\ldots.,\; m.$$

$$L_1 = (m-n+k-l+i),\quad L_2 = (n+l+j),$$

$$I_\alpha = \frac{(2k+i+j+1)!\ (2L_1+2\delta_{1\alpha})!\ (2L_2+2\delta_{2\alpha})!}{(k-m)!\ \ (L_1+\delta_{1\alpha})!\ \ (L_2+\delta_{2\alpha})!} \quad (\alpha=1,2) \qquad (2.49)$$

The relation between A and C coefficients in Eq. (2.47) can be summarized in a matrix form :

$$\underset{N\times 1}{\{ A \}} = \underset{N\times N}{[B]} \underset{N\times 1}{\{ C \}} \qquad (2.50)$$

where N is total number of coefficients A or C. For an incomplete polynomial , the number of terms depend not only on the parameter M of Eq. (2.48) but also on parameters i and j of Eq. (2.45).

2.5 STRESS INTENSITY FACTORS

Once the coefficients C are determined by solving Eq. (2.50) for given loading on the crack surface, the stress intensity factors corresponding to these loads are computed from the following Eq. (19) for Mode I problem

$$K_I = 8\ G \left(\frac{\pi}{a_1 a_2}\right)^{1/2} A^{1/4} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{M} \sum_{l=0}^{k} (-2)^{2k+i+j}$$

$$(2k+i+j+1)!\ \frac{1}{a_1 a_2} \left(\frac{\cos\theta}{a_1}\right)^{2k-2l+i} \left(\frac{\sin\theta}{a_2}\right)^{2l+j} C^{(ij)}_{3,k-l,l} \qquad (2.51)$$

where θ is elliptic angle and

$$A = a_1^2 \sin^2\theta + a_2^2 \cos^2\theta \qquad (2.52)$$

For Mode II and Mode III equations for stress intensity factors could be found in (19,20).

2.6 NUMERICAL EVALUATION OF INCOMPLETE ELLIPTIC INTEGRALS OF FIRST AND SECOND KIND

Since evaluation of incomplete elliptic integrals of first and second kind is necessary to elliptic integral evaluation of Eq. (2.36), the Arithmetic-Geometric Mean method and Landen's Descending Transformation (28) are to be discussed . The parameter k or k', and crack size a_1, a_2 are used to calculate parameters m and α by following (28) Eqs.;

$$m = k^2 \; ; \; \alpha = \sin^{-1}(\sqrt{m}) \; ; \qquad (2.53)$$

These parameters are now used to determine (a_0, b_0, c_0) to start the process of Arithmetic-Geometric Mean Method in which one proceeds to determine the number triples (a_1, b_1, c_1), $(a_2, b_2, c_2), \ldots , (a_N, b_N, c_N)$ according to following scheme of arithmetic and geometric means.

$$a_0 = 1 \qquad b_0 = \cos\alpha \qquad c_0 = \sin\alpha \qquad (2.54)$$

$$a_1=(a_0+b_0)/2 \qquad b_1=(a_0 b_0)^{1/2} \qquad c_1=(a_0-b_0)/2$$

$$a_2=(a_1+b_1)/2 \qquad b_2=(a_1 b_1)^{1/2} \qquad c_2=(a_1-b_1)/2$$

$$a_N=(a_{N-1}+b_{N-1})/2 \qquad b_N=(a_{N-1} b_{N-1})^{1/2} \qquad c_N=(a_{N-1}-b_{N-1})/2 \quad (2.55)$$

This process is terminated at the Nth step when c_N becomes sufficiently small to the degree of accuracy required. In present work converging value of $c_N = 10^{-7}$ is taken. One can now determine incomplete elliptic integral of fist kind ie., $F(u_1)$ and of second kind ie., $E(u_1)$ using following relationships.

$$u_1 = F(u_1) = \phi_N / (2^n a_N) \qquad (2.56)$$

$$E(u_1) = \sum_{i=1}^{N} c_i \sin \phi_i + Ec . F(u_1)/Kc \qquad (2.57)$$

where Kc = complete elliptic integral of first kind.

Ec = complete elliptic integral of second kind and are given by

$$Kc = \pi / (2 a_N) \qquad (2.58)$$

$$Ec = Kc + Kc [1/2 \sum_{i=0}^{N} 2^i c^2{}_i] \qquad (2.59)$$

Angles ϕ_i in Eq. (2.56) and (2.57) are determined using following Landen's descending Transformations.

$$\tan (\phi_{n+1} - \phi_n) = (b_n/a_n) \tan \phi_n, \ \phi_0 = \sin^{-1}(sn) \qquad (2.60)$$

This procedure of inverting Jacobian elliptic function and determining desired elliptic integrals is implemented in the program. Particular caution is needed in implementing Eq. (2.60) since $\tan^{-1}$ ($\tan (\phi_N)$) could be considerably off the true value ϕ_N if proper correction is not applied to the computed value.

2.7 FINITE ELEMENT ALTERNATING METHOD

The finite element alternating technique as used in present work alternately uses following two solutions;

1.Solution to the elliptical crack subjected to arbitrary normal and shear loading on the crack surface , in an infinite solid as discussed in sections 2.2 to 2.5.

2. A general solution technique ie three dimensional finite element method as briefed in section 2.1.

The basic concept of alternatingly using these two solution procedures to give desired solution to the problem of a finite plate containing crack is presented in Fig.2.2. The solution of one method is used to find load for the other method for example FEM solution of an uncracked body is used to find stresses on the crack face, which are reversed and used as loads on the crack face of an infinite body. This is done in order to satisfy zero stress condition on the crack surface. Solution 1. is then used to determine stresses at the boundary of the finite body in consideration, these stresses are reversed and used as load for next FEM solution. The FEAT requires following steps as shown in Fig. 2.3.

(1) Solve the uncracked body under the given external loads by using FEM. The uncracked body has the same geometry with the given problem except the crack.

(2) Using finite element solution, stresses on the crack surface of original crack are computed. When the solution has converged stresses at the crack surface should be negligible, otherwise there would be some stresses still on the crack face. These stresses are termed residual stresses. Residual stresses in iteration two of this method are shown in Fig. 2.4.

$\sigma_{33}^{(1)}, \sigma_{33}^{(2)}$ FEM Solutions ; $\sigma_{33}^{(1)R}, \sigma_{33}^{(2)R}$ Reverse of stresses $\sigma_{33}^{(1)}, \sigma_{33}^{(2)}$;

$\sigma^{(1)}, \sigma^{(2)}$ from step 7.

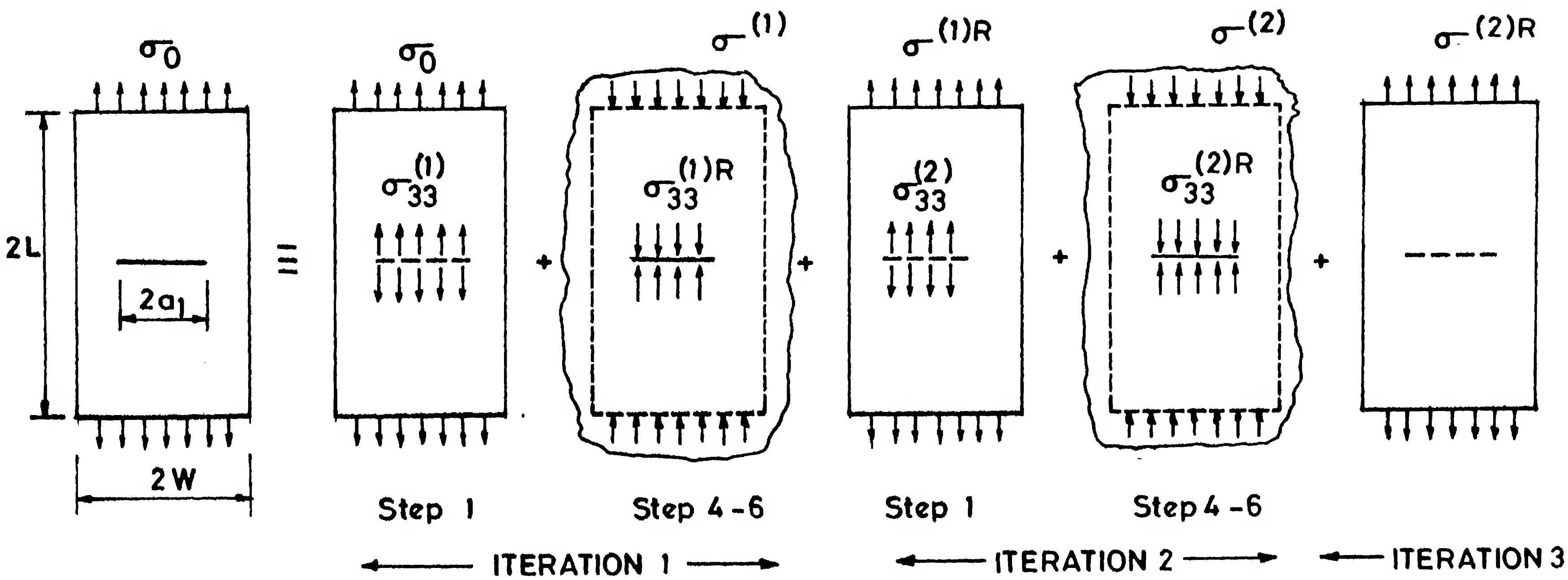

FIG. 2.2 FEAT APPROACH : SUPERPOSITION OF VARIOUS CONFIGURATIONS

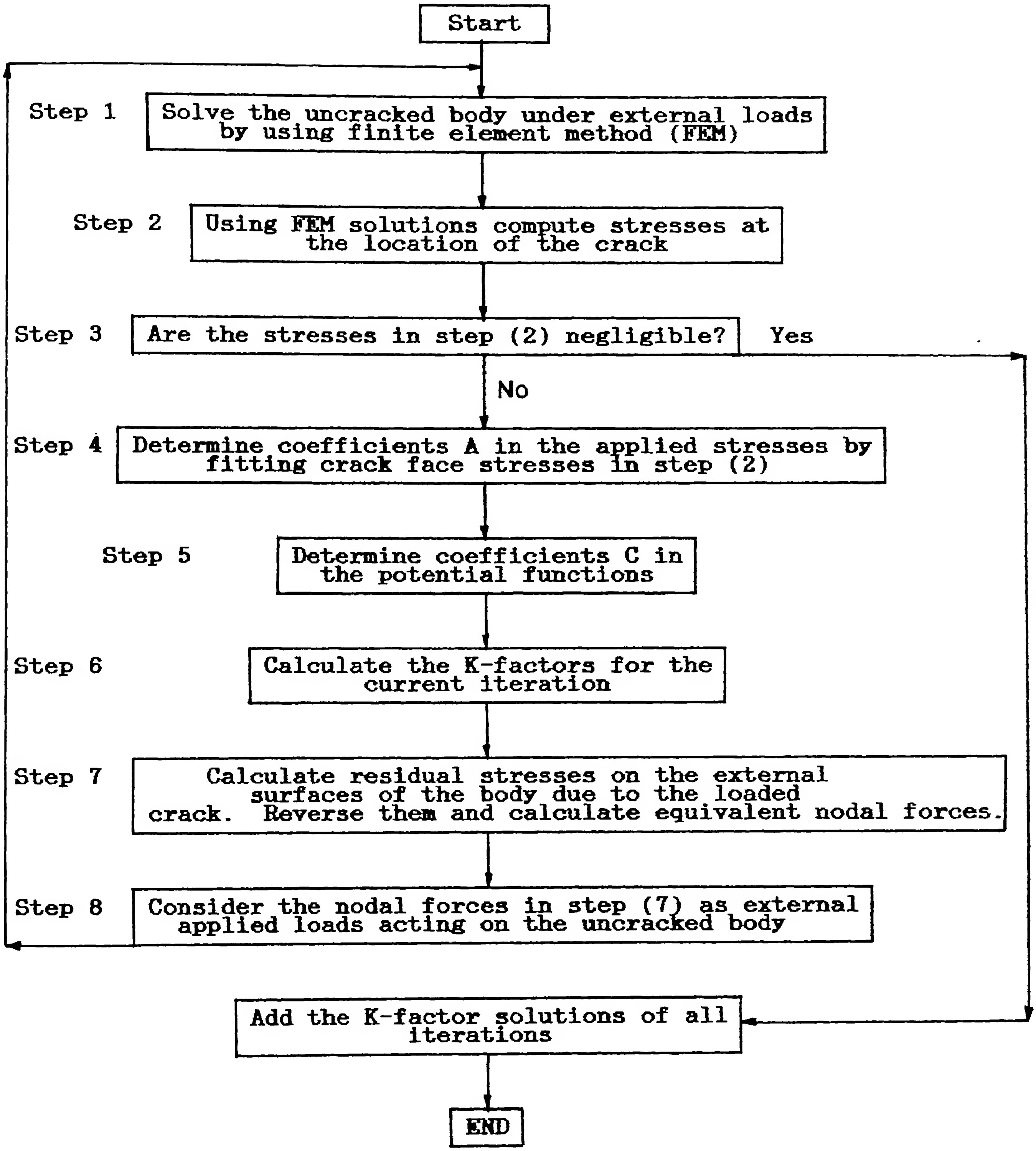

Fig. 2.3 Flow chart for finite element-alternating technique

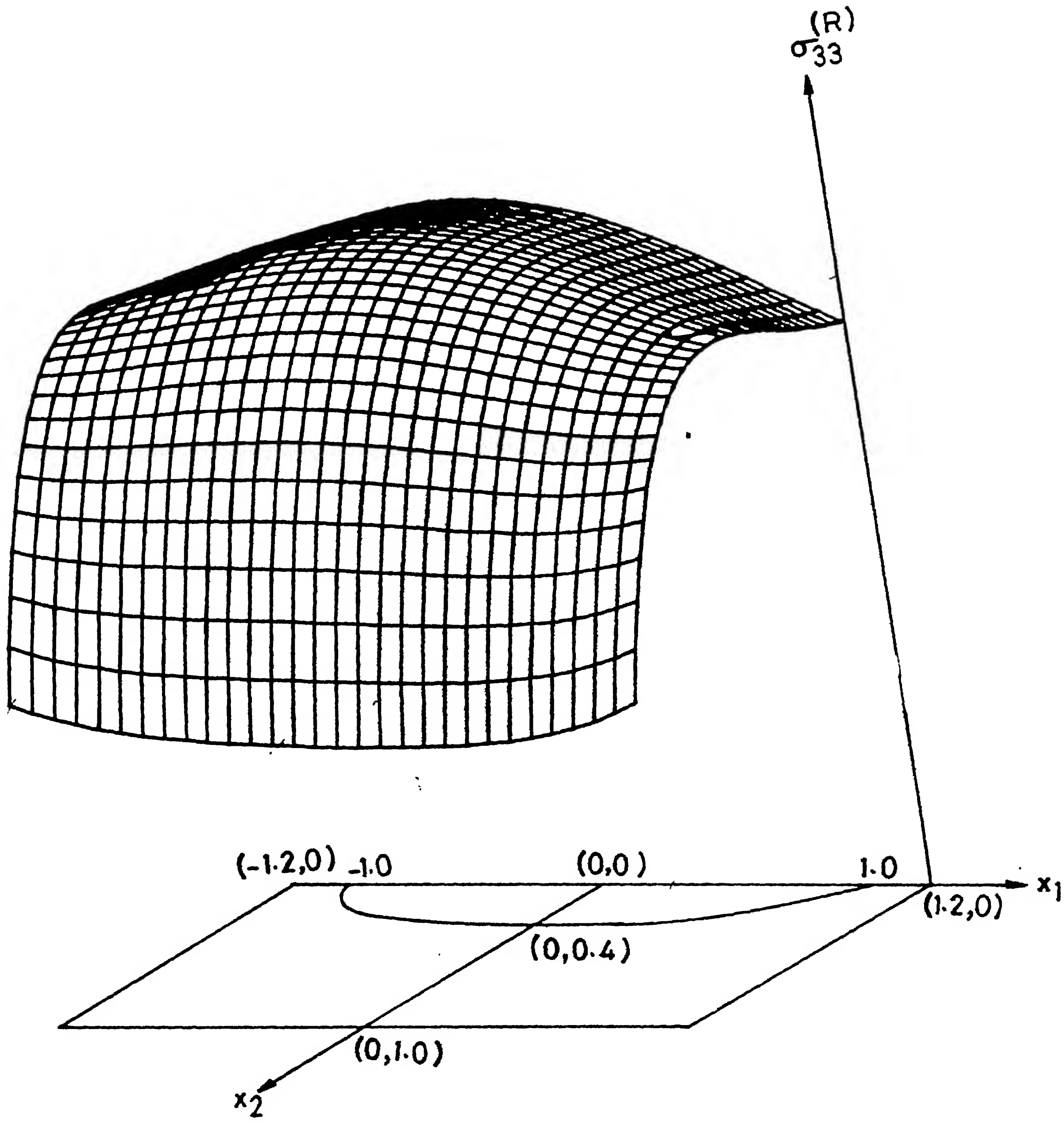

FIG. 2.4 RESIDUAL STRESSES IN $X_1 X_2$ PLANE

(3) Compare the residual stresses in step (2) with a permissible stress magnitude. In the present study three percent of the maximum external applied stress is used for the permissible stress magnitude.

(4) To satisfy the stress stress boundary condition, reverse the residual stresses. Then determine coefficients A of Eq. (2.45) by using following least squares method:

$$I_\alpha = \int_{Sc} (\sigma_{3\alpha}^{(R)} - \sigma_{3\alpha}^{(\phi)})^2 \, dS \quad (\alpha = 1,2,3) \tag{2.61}$$

where $\sigma_{3\alpha}^{(R)}$ is the reversed residual stresses calculated by the finite element solution, $\sigma_{3\alpha}^{(\phi)}$ is defined by Eq. (2.45), Sc is the region of the crack, and I_α are the functionals to be minimized.

Rewriting Eq. (2.45) in matrix form :

$$\sigma_{3\alpha}^{(\phi)} = \{ L \}^T \{ A_\alpha \} \tag{2.62}$$

and substituting Eq. (2.62) in Eq. (2.61), following relation between A coefficients and the residual stresses is obtained.

$$\{ A \} = [H]^{-1} \{ R_\alpha \} \tag{2.63}$$

where

$$[H] = \int_{Sc} \{ L \} \{ L \}^T \, dS \tag{2.64}$$

$$\{ R_\alpha \} = \int_{Sc} \{ L \} \sigma_{3\alpha} \, dS \tag{2.65}$$

(5) Determine the coefficients C of Eq. (2.46) in the potential function by solving Eq. (2.5Ø).

(6) Calculate the stress intensity factors for the current iteration by substituting C coefficients in Eq. (2.51, 2.52).

(7) Calculate the residual stresses on the external surfaces of the body due to loads in step (4) using solution 1 ie Eq. (2.17a -2.17f). To satisfy the stress conditions of original problem , reverse the residual stresses and calculate equivalent nodal forces. This step is highly computer time intensive and takes considerable computational time. These nodal forces { Q } can be expressed in terms of coefficients C :

$$\{ Q \}_m = - [G]_m \{ C \} \qquad (2.66)$$

and

$$[G]_m = \int_{Sm} [N]^T [n] [p] \, dS \qquad (2.67)$$

where m denotes the number of finite element,

[N] is the matrix of isoparametric element shape functions,

[n] is the matrix of normal direction cosines and

[p] is the basis function matrix for stresses and can be derived from Eq. (2.17a - 2.17f). For the Mode I problem in hand this has been derived as:

$$
[\,p\,] = \begin{bmatrix}
\sum_{k=0}^{M} \sum_{l=0}^{k} 2G\,(F_{kl,11}+F_{kl,22}+x_3\,F_{kl,311}) \\
\sum_{k=0}^{M} \sum_{l=0}^{k} 2G\,((1-2v)F_{kl,12}+x_3\,F_{kl,312}) \\
\sum_{k=0}^{M} \sum_{l=0}^{k} 2G\,x_3\,F_{kl,313} \\
\sum_{k=0}^{M} \sum_{l=0}^{k} 2G\,(F_{kl,22}+2vF_{kl,11}+x_3\,F_{kl,322}) \\
\sum_{k=0}^{M} \sum_{l=0}^{k} 2G\,x_3\,F_{kl,323} \\
\sum_{k=0}^{M} \sum_{l=0}^{k} 2G\,(-F_{kl,33}+x_3\,F_{kl,333})
\end{bmatrix} \qquad (2.68)
$$

In order to save computational time, the matrix $[\,G\,]_m$ for all elements on the boundary of the body is calculated prior to the start of the iteration shown in Fig. 2.2.

(8) Consider the nodal forces in step (7) as external applied loads acting on the uncracked body.

All steps in the iteration process are repeated until the residual stresses on the crack surface become negligible (step 3). Thus the solution for the given problem is the superposition of all the solutions obtained thus for and hence the stress intensity factors of all iterations are added to obtain desired solution.

CHAPTER III

PROBLEM FORMULATION AND METHODOLOGY

3.1 Introduction

This chapter describes the details of Finite Element Alternating Technique as applied to the problem of a plate, containing an embedded crack, under unidirectional tension. A three dimensional technique needs to be used here as plate is of finite thickness. The program structure and techniques for minimizing CPU time have also been explained.

3.2 Problem description

Stress intensity factors along the crack boundary of an embedded crack are determined using FEAT. The embedded crack is elliptic and is at the center of a finite sized plate (refer fig 3.1).

The problem is analyzed using FEAT under the following assumptions.

(i) Plate material is isotropic and linearly elastic.

(ii) Small deformation theory is applicable for the purpose of calculating stresses using Finite Element Method.

(iii) Plate is subjected to normal uniform stress loading on surfaces $x_3 = \pm L$ (fig 3.1), thus the stresses at the crack front are normal and hence all relations in FEAT are for Mode I.

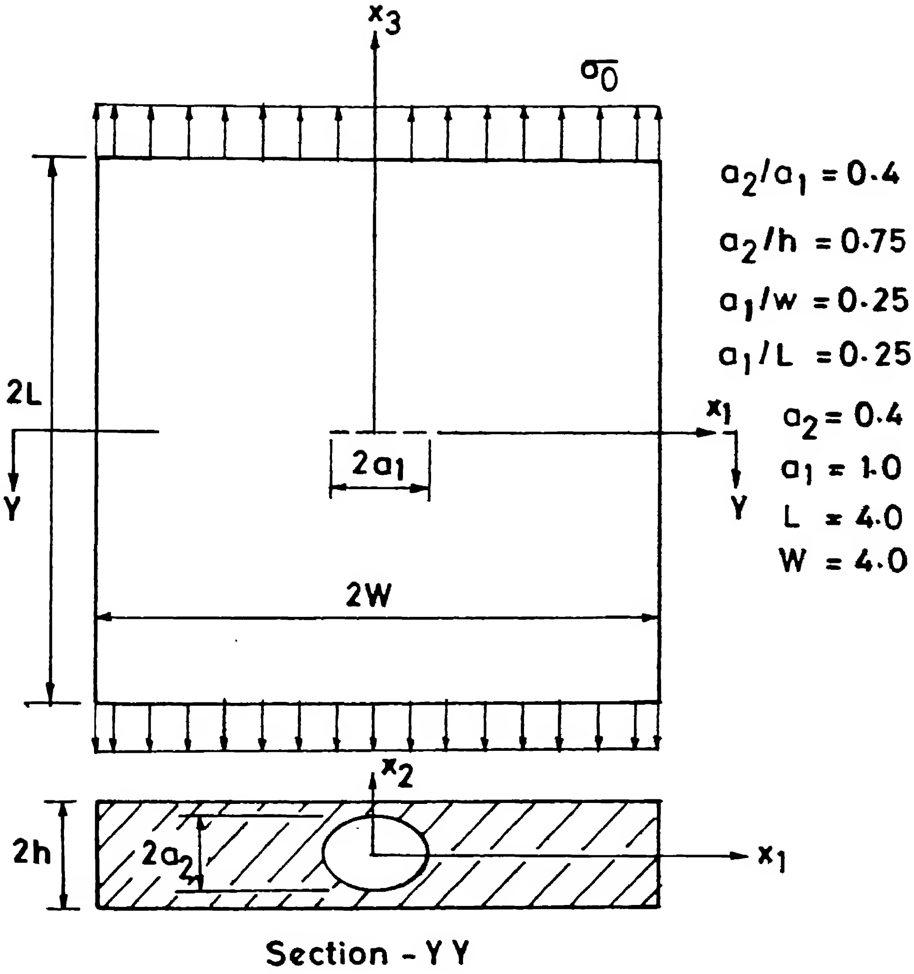

FIG. 3.1 ELLIPTICAL CRACK IN A FINITE THICKNESS PLATE

3.3 FINITE ELEMENT MESH

The finite element discretisation of the uncracked body has considerable influence on convergence of the technique as well as accuracy of the solution. Fig. 3.2 shows the finite element mesh of the uncracked body. Full advantage is taken of reflective symmetry in three directions and only one eighth of the plate has been modeled. Fig. 3.2 also shows that crack geometry is modeled by curved edged elements instead of straight edged elements as shown in Fig. 3.3, Though manual input data preparation is relatively more involved with curved edged elements, following major advantage exists in using curved edged elements. In case of curved edged elements only six terms are needed in the polynomial of Eq. (2.45) for describing the load on the crack face, as against twenty one terms in case of straight edged elements (1) in order that satisfactory convergence is obtained. This implies that $[G]_m$ matrices in Eq. (2.66) (which are calculated for each element on the external boundary of the body), be 60 x 6 instead of 60 x 21 when straight edged elements are used. It can be said that for computation of these matrices if T CPU seconds are required by straight edged elements then only T/3 seconds or less would be needed by curved ones. Thus curved edged elements reduce the CPU time for computation of $[G]_m$ matrices to less than one third.

3.4 PROGRAM STRUCTURE

While developing the program attempts have been made to modularise entire calculations in subroutines and functions so that following objectives are fulfilled to the highest possible extent.

(i) Unnecessary repetition of Calculations is avoided. This is very important to the efficiency of the program.

(ii) Program build up is easy and each module could be separately tested and then assembled into the whole program.

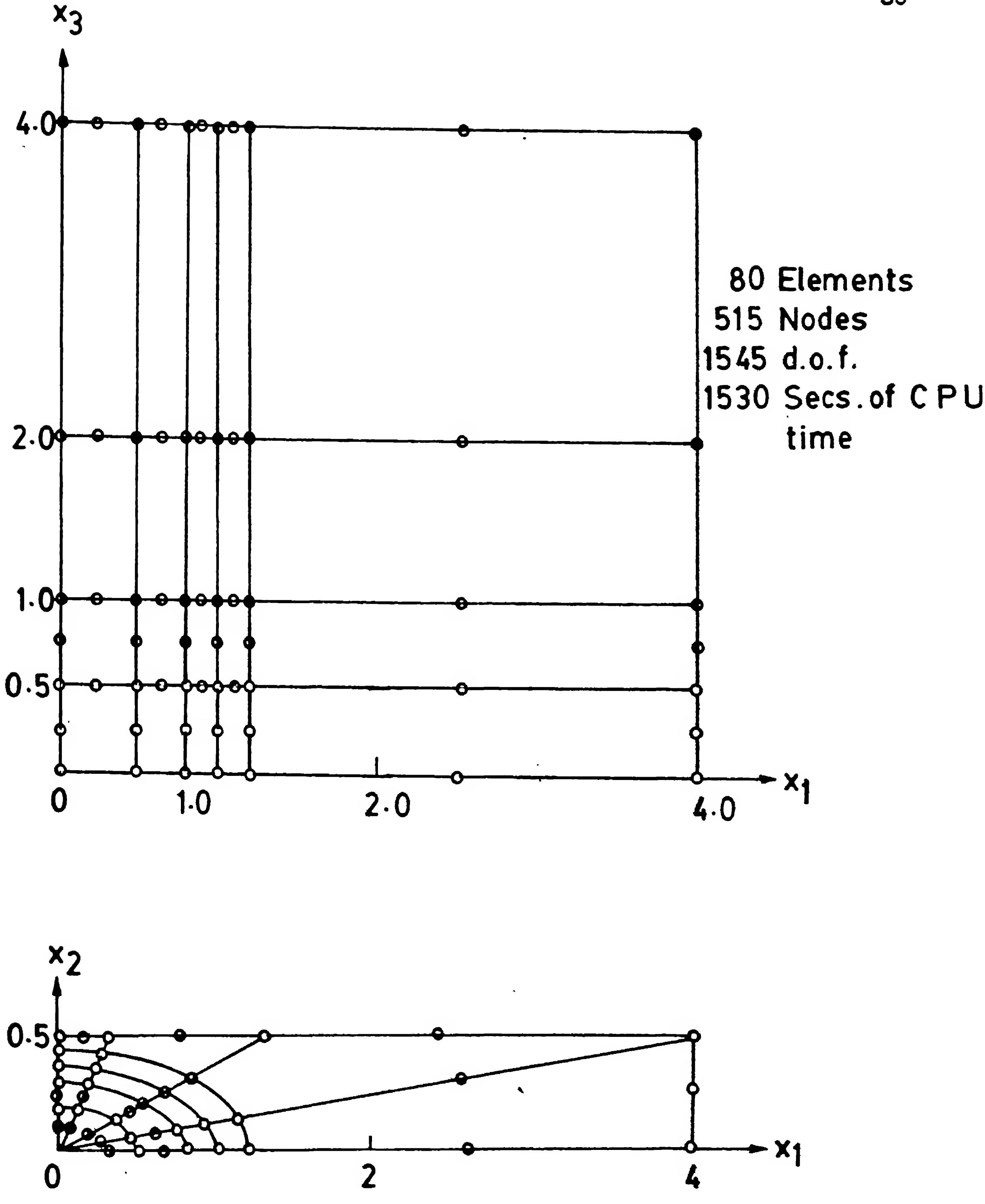

FIG. 3.2 FEM MESH USING CURVED EDGED ELEMENTS

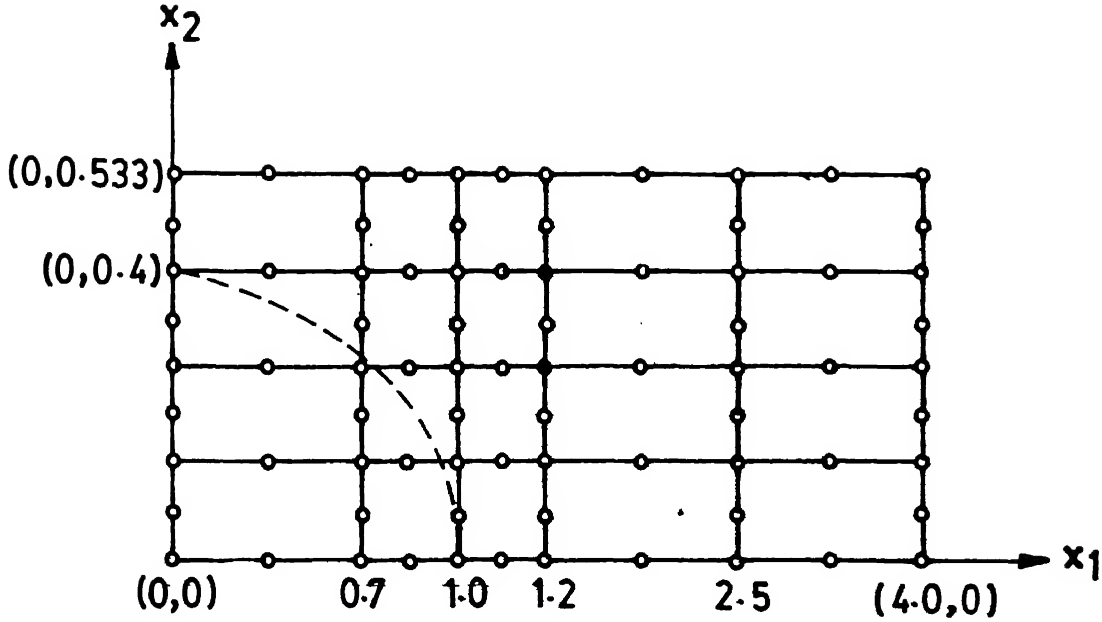

FIG.3.3 FEM MSEH USING STRAIGHT EDGED ELEMENTS

With this modularisation concept program was basically divided in two parts, one for FEM and the other for solution to the loaded crack problem referred to as FEAT. The FEM portion starts with subroutine FEM3D and all those called by FEM3D. This portion of the structure is given in Fig.3.4. It should be noted that coordinates of all the midside nodes lying on straight edges are interpolated by the program.

Program structure of FEAT part of the program is shown in Fig. 3.5 and 3.6. Program structure clearly shows that suitable modules have been made that could easily be used and called by different subroutines.

Computation of generic elliptic integrals of Eq. (2.36) is done for each of the 32 boundary elements, at each surface on boundary, at each of the 49 Gauss points and for each element of the P matrix calculation ie. more than 57,600 times per element. Much higher number of factorial or Gamma function evaluation is done. It is important that these computations are done in least possible time. For this purpose Elliptic integral calculations are split in two subroutines. One subroutine precalculates integral arrays of equations (2.38), (2.40) and (2.41) for different p, q, and r parameters for each Gauss point in consideration. Another subroutine now simply applies Eq. (2.37) using precalculated array of integrals to get the desired generic elliptic integral.

It is worth noting that argument of factorial function does assume negative values in Eq (2.35) and for these, the corresponding function subroutine is not called at all instead following identity (28) for factorial of negative integer is taken.

$$1 / n! = 0 \; ; \; n < 0. \qquad (3.1)$$

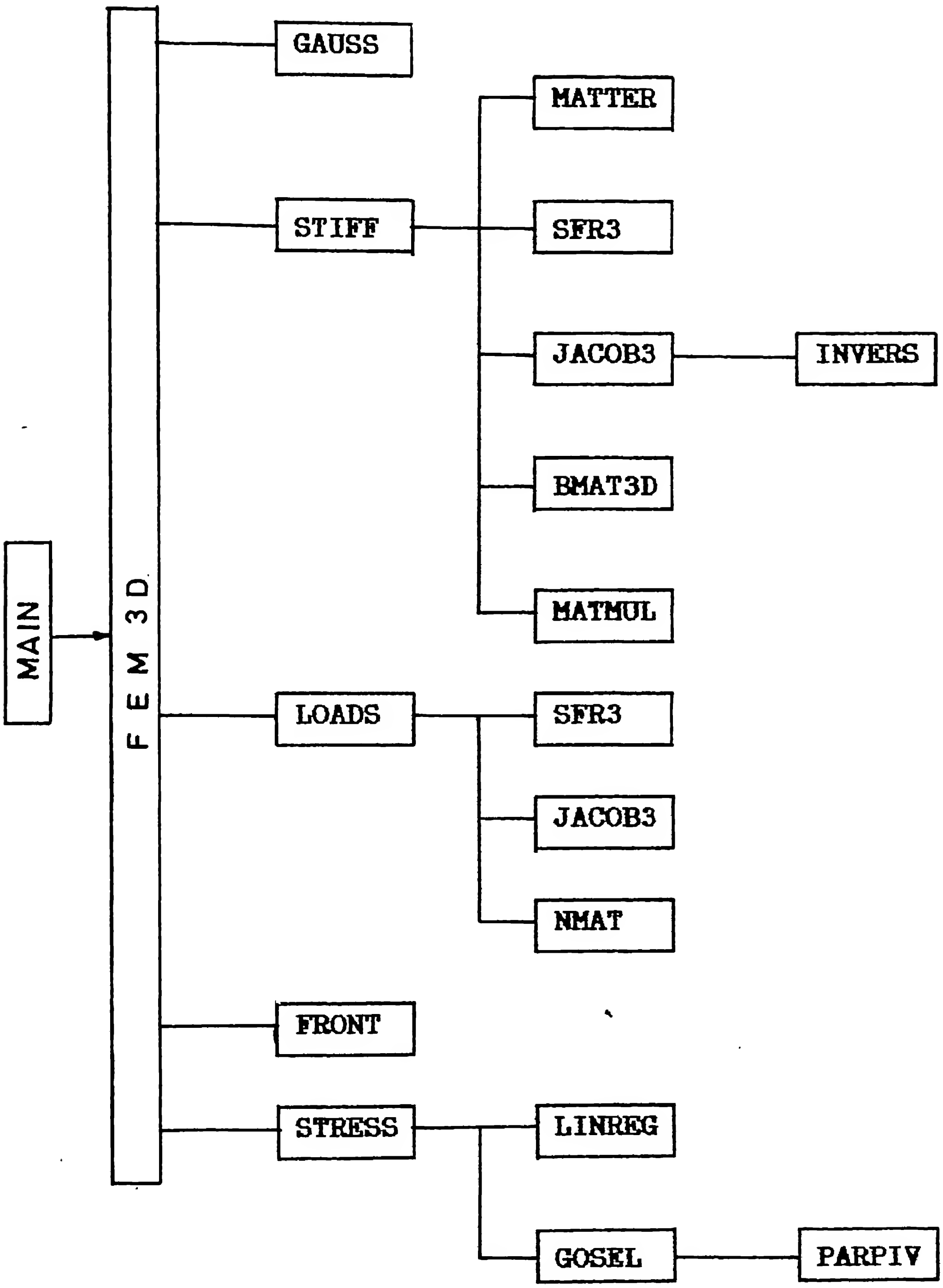

Fig. 3.4 Program structure of FEM3D

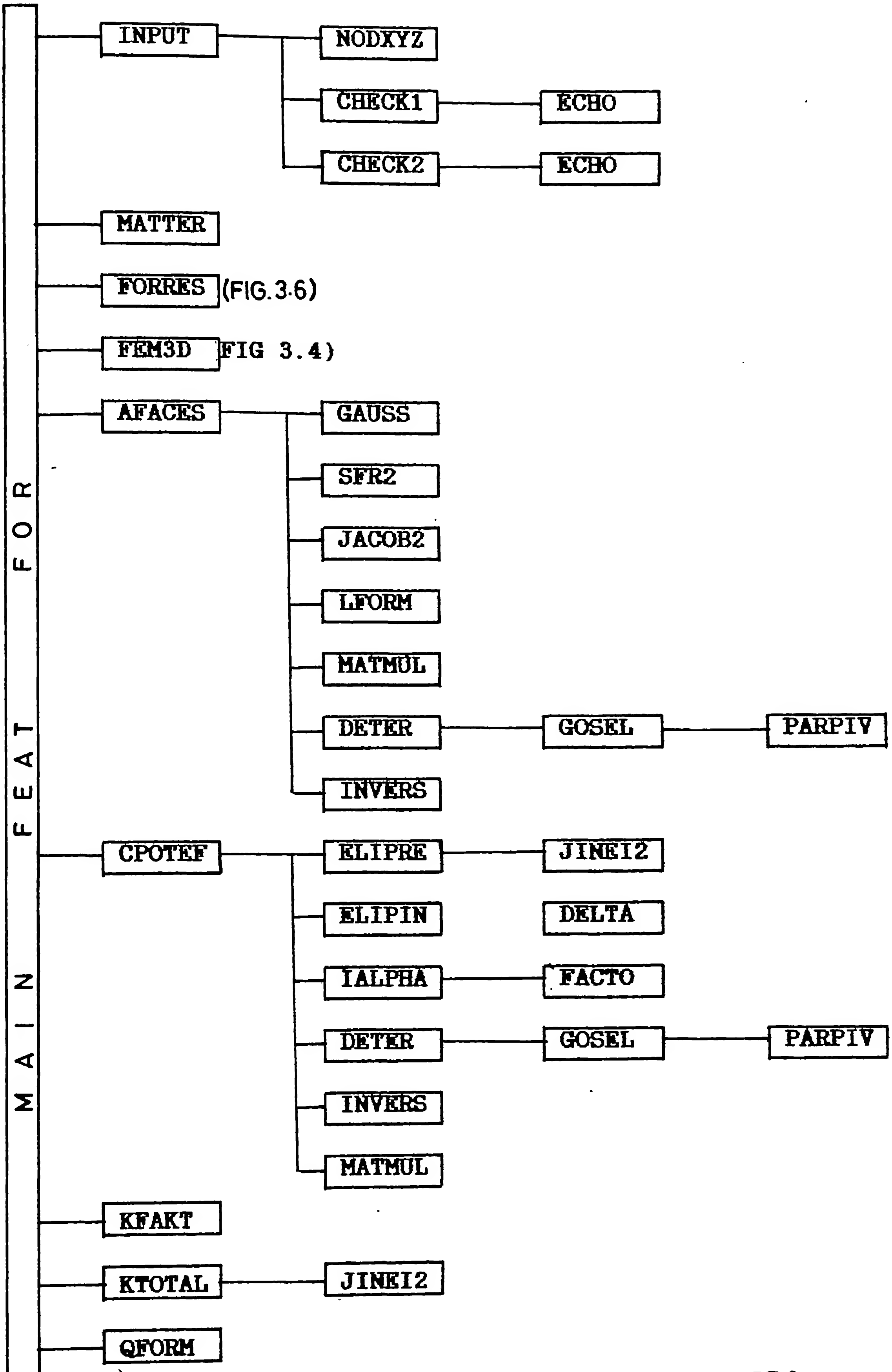

Fig. 3.5 Program structure of FEAT PART 1.

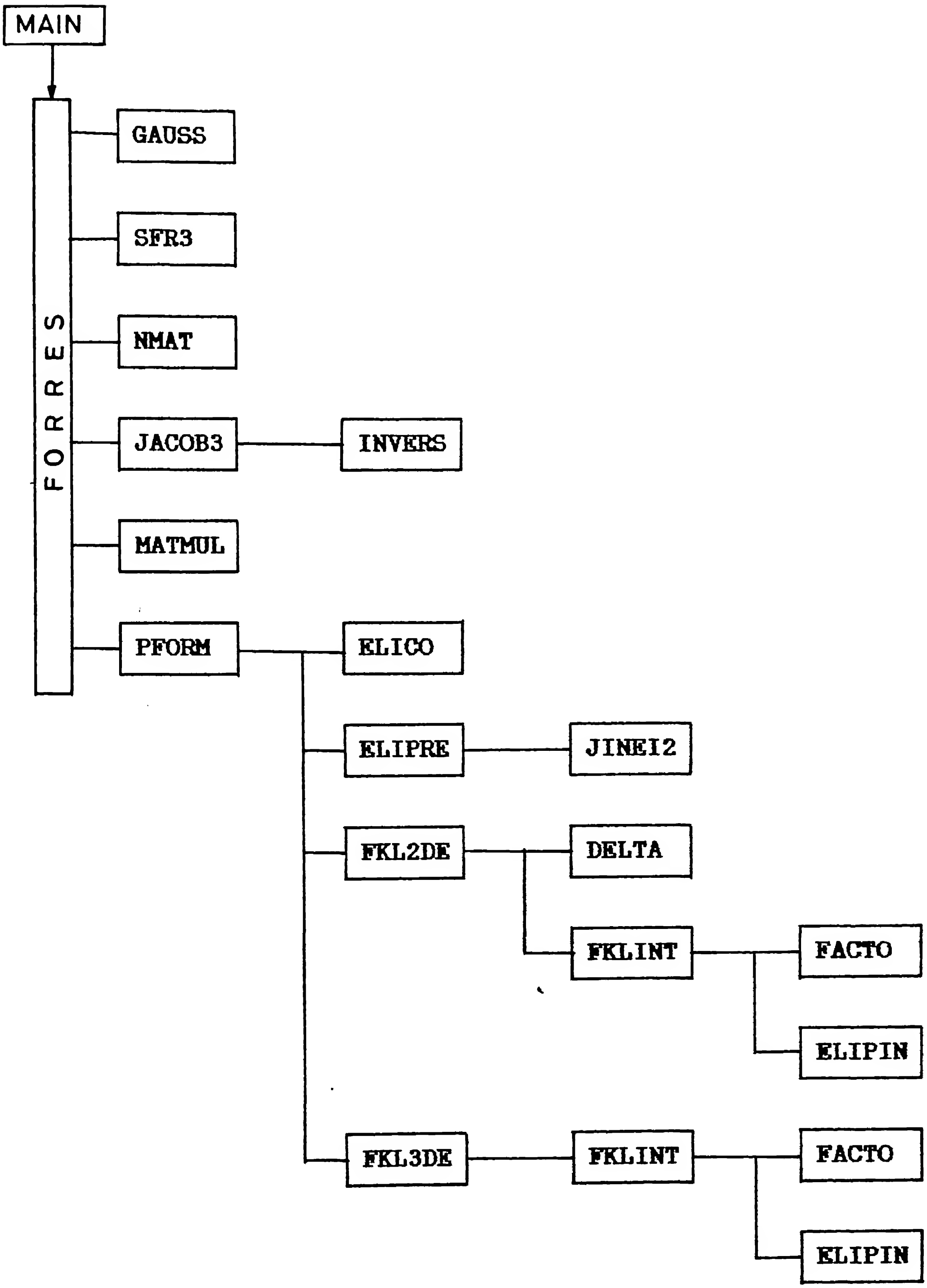

Fig. 3.6 Program structure of FEAT PART II.

Computation of ellipsoidal coordinates Φ_1, Φ_2, and Φ_3 ie. roots of polynomial of Eq (2.20) are repeatedly performed for each Gauss point for all boundary surfaces. This is done by the program using Graeffe's root squaring method.

3.5 Testing Elliptic Integral Subroutines

Evaluation of each module is significant and leads to considerable saving in program development. Testing of mainly modules relating to elliptic integral evaluation need to be briefed here.

These subroutines have been tested for following crack related data and results have been compared with (20) in Fig. 3.7 for parameters l, m, and n given by

$$a_1 = 2,\ a_2 = 1,\ a_3 = 0,\ \Phi_1 = -4,\ \Phi_2 = -1,\ \Phi_3 = 0.0625:$$

$$l = p - q,\ m = q - r,\ n = r \tag{3.2}$$

It is clear from Fig. 3.7 that the values calculated by program match well with those of ref. (20).

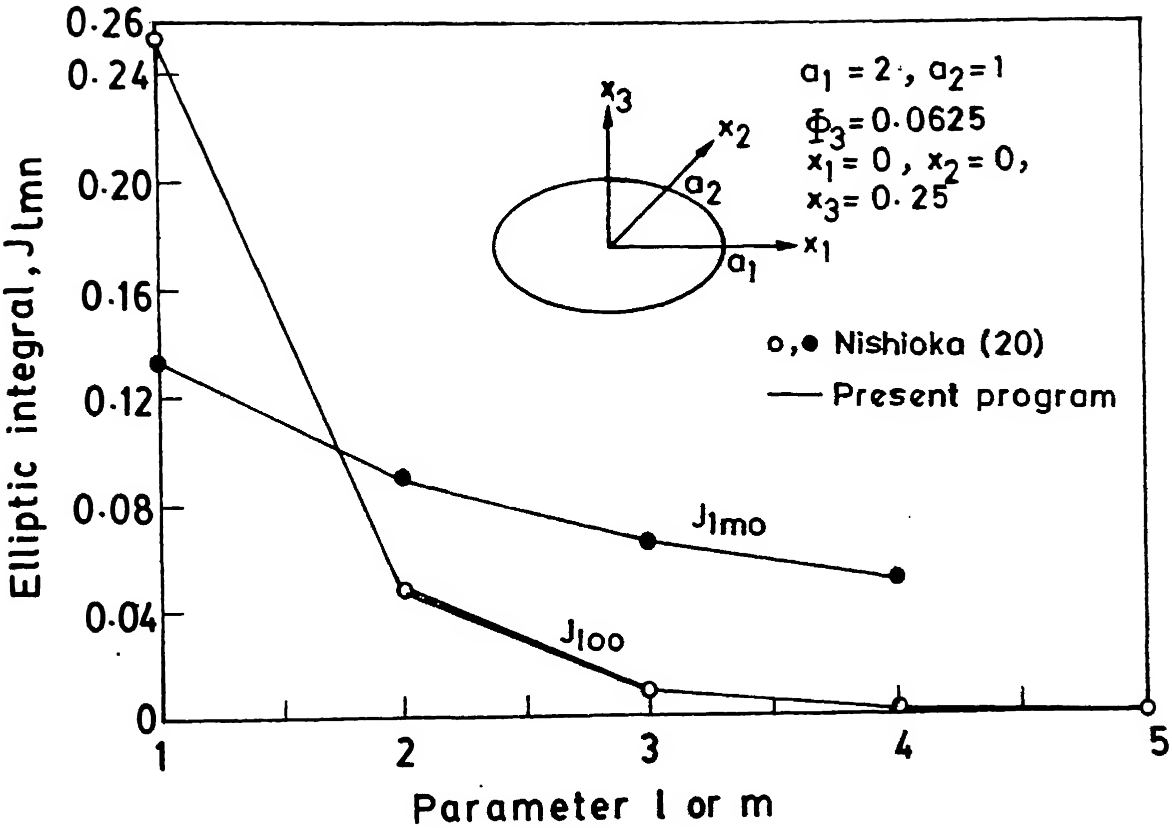

FIG. 3.7 ELLIPTIC INTEGRALS OBTAINED BY PROGRAM FEAT

CHAPTER IV

RESULTS AND DISCUSSION

4.1 INTRODUCTION

This chapter presents analysis of the results obtained by using the FEAT program developed. Results are compared with the solution for the problem of finite thickness plate containing an embedded crack (20).

The crack and plate geometry is as shown in Fig. 3.1. Plate thickness 2h to crack parameter $2a_2$ ratio of 0.75 is taken. The aspect ratio of crack (a_2 / a_1) is taken to be 0.4. Plate material is considered to have following properties

Young's modulus, E = 1000 Mpa
Poisson's ratio = ν = 0.3

FEM discretisation of the plate is as shown in Fig. 3.2. The mesh consists of 80 elements and 515 nodes and 1545 total degrees of freedom. There are four layers of 20 elements each. Plate size is also shown in Fig. 3.2. In the frontal technique Maximum FRONT WIDTH of 309 is encountered.

In this mesh 32 elements are at the boundary of the mesh (one eighth of the plate). For these elements $[\ G\ ]_m$ matrices are calculated at the start of the iteration process. Numerical integration over each of the surfaces of these elements is done using 7 x 7 Gauss points as the terms of $[\ G\ ]_m$ involve polynomial of 13^{th} degree. The loading on the crack face is approximated using six terms ($1,\ x_1^2,\ x_2^2,\ x_1^2 x_2^2,\ x_1^4,\ x_2^4$) of a fourth order polynomial of Eq (2.45) . The parameter M, i, and j in Eq (2.45) for polynomial loading are taken to be M = 2; i = 0, j = 0. In numerical integration of Eq (2.64) and (2.65) 5 x 5 Gauss points were used for each element lying on crack face.

4.2 Stress Intensity Factors

The method has shown monotonic convergence and in four iterations the solution converged to the true answer. Convergence is checked using the normalized residual force on the crack face, which is given by

$$\frac{\int_{S_c} \sigma_{33} \, dS}{\sigma_0 \pi a_1 a_2 / 4.0}$$

The term in the denominator represents the force on the crack face, if it were subjected to the same uniform distributed load as that acting on the plate end. In four iterations monotonic reduction of normalized residual forces on the crack face is shown in Fig. 4.1, Table 4.1. In last iteration this nondimensional, normalized force has a value of 0.028 which is satisfactory. This could be further lowered to 0.01 by changing the algorithm as would be discussed in next chapter.

Fig. 4.2 , Table 4.2 shows convergence of stress intensity factors with successive iterations. Fig. 4.2 also shows variation of stress intensity factors K_I along crack boundary w.r.t. angle θ at the end of iterative process. It is important to note that stress intensity factors are higher at points nearer to the plate surfaces, ie. at angles θ = 90°. This trend is similar to stress intensity factors at crack periphery of an elliptic crack in an infinite plate subjected to tension as shown in Table 4.2. It is only by normalizing the stress intensity factors of finite plate with those of an infinite plate, that a much clearer picture emerges. The variation of normalized stress intensity factors is shown in Fig. 4.3, Table 4.3. It is clear from the figure that the normalized stress intensity factors are higher at θ° = 90°, which leads to the conclusion that effect of

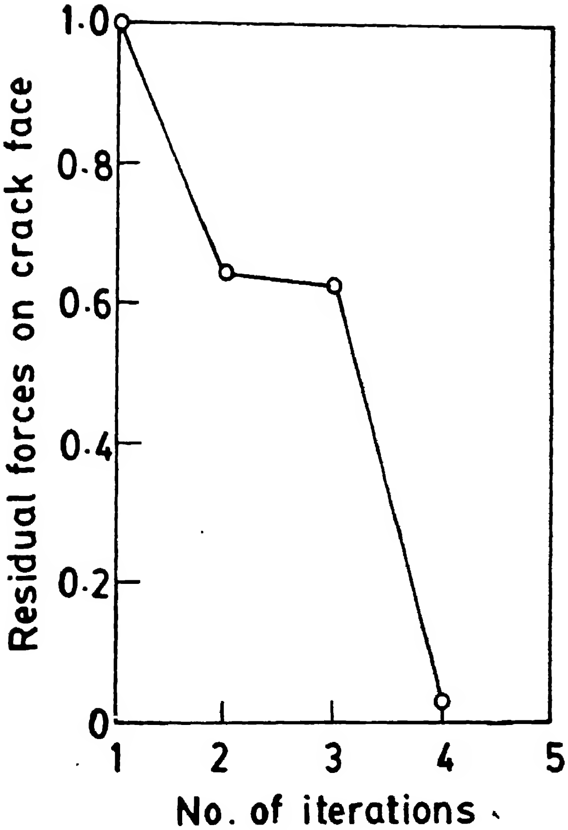

FIG. 4.1 NORMALISED RESIDUAL FORCES ON CRACK FACE WITH SUCCESSIVE ITERATIONS

Table 4.1, Normalized Residual Force On Crack Face

Iteration Number	Normalized Residual Force on crack face
1	Ø.997
2	Ø.64Ø
3	Ø.63Ø
4	Ø.Ø28

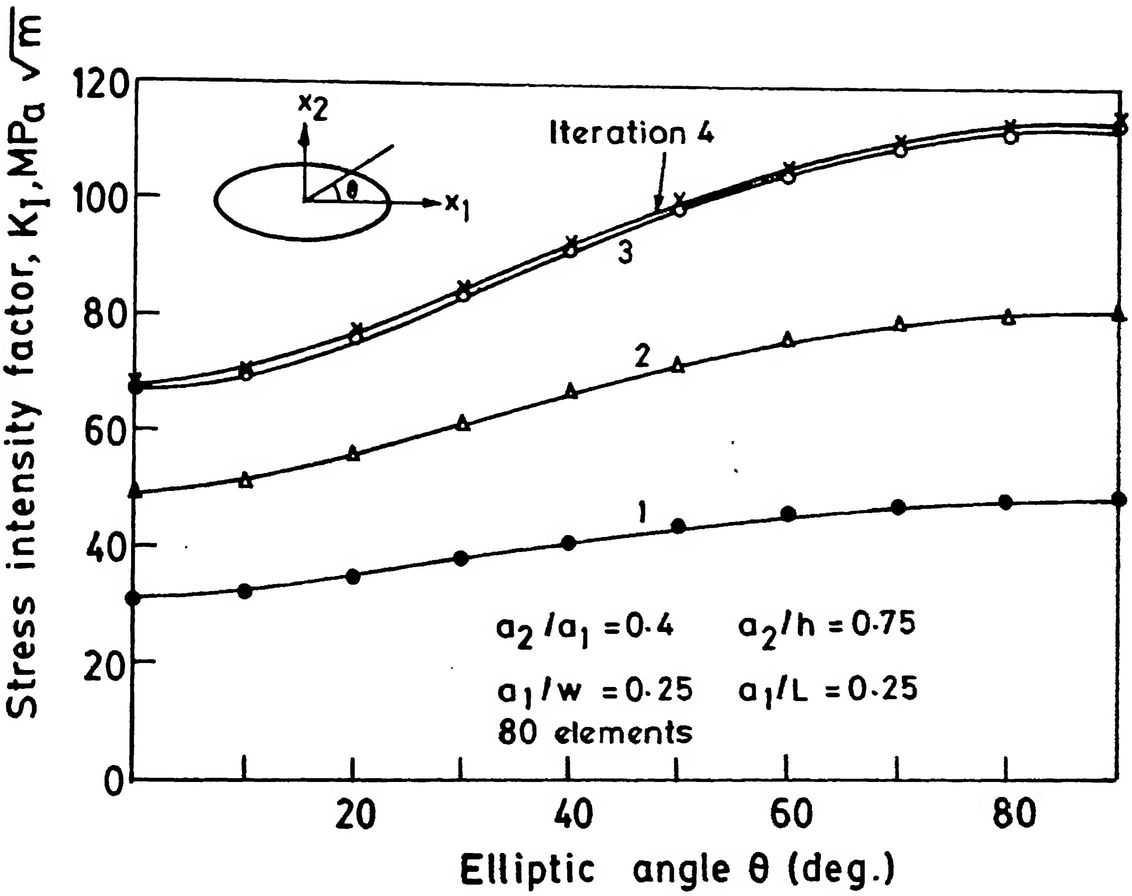

FIG. 4·2 CONVERGENCE OF STRESS INTENSITY FACTORS WITH SUCCESSIVE ITERATIONS

Table 4.2, Stress intensity factors

Elliptical	Stress intensity factors K_I, MPa				
angle θ°	Iter 1	Iter 2	Iter 3	Iter 4	Infinite Plate
0	30.78	48.90	66.51	67.29	61.62
10	31.93	50.82	69.16	69.98	64.52
20	34.69	55.47	75.64	76.54	69.46
30	37.96	61.10	83.58	84.58	76.00
40	41.07	66.66	91.51	92.62	82.21
50	43.75	71.62	98.68	99.89	87.56
60	45.88	75.70	104.68	105.96	91.83
70	47.42	78.74	109.14	110.50	94.93
80	48.35	80.60	111.92	113.32	96.78
90	48.66	81.23	112.85	114.27	97.40

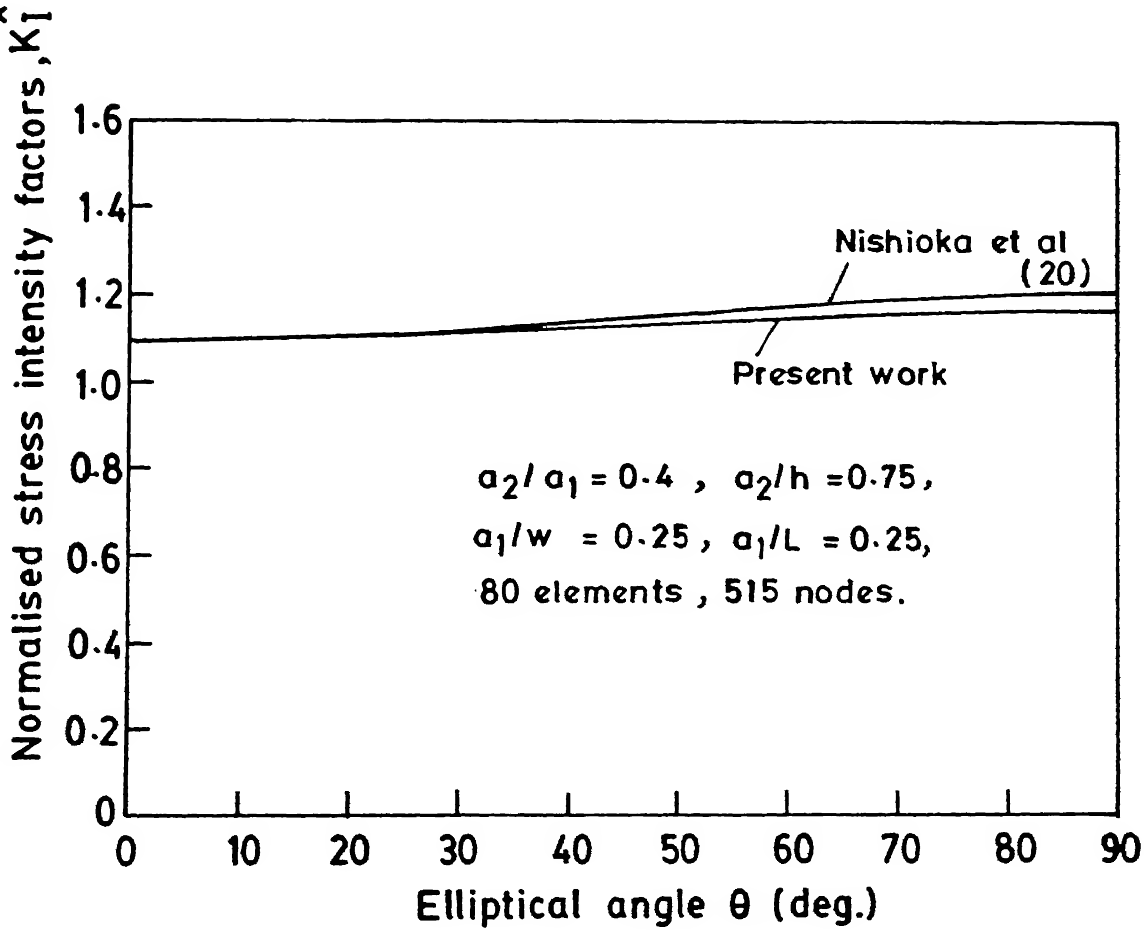

FIG. 4.3 NORMALISED STRESS INTENSITY FACTORS, A COMPARISON

$$K_I^* = K_I \Big/ \left[\frac{\sigma_0}{E(K)} \cdot \sqrt{\pi a_2/a_1}\; (a_1^2 \sin^2\theta + a_2^2 (M^2\theta)^{1/4} \right]$$

Table 4.3: Normalized Stress Intensity Factors

Elliptic Angle $\theta°$	Normalized Stress Intensity Factors Present Work	Nishioka (20)
0	1.092	1.10
10	1.095	1.10
20	1.102	1.105
30	1.113	1.12
40	1.127	1.14
50	1.141	1.16
60	1.154	1.18
70	1.164	1.20
80	1.171	1.208
90	1.173	1.21

finite plate thickness of the plate is more at the points, nearer to the plate surfaces. Fig. 4.3 and Table 4.3, also compare normalized stress intensity factors obtained in present work with those of Nishioka et al.(20). Since the maximum error of 3.1 % occurs at θ = 90°, it can be said that results compare well with those of Nishioka. The error could however be due to use of only linear regression analysis in step (2) of FEAT (Fig. 2.3). Nishioka may have instead used Multiple polynomial regression for this purpose.

The program was developed and finally run on ND-500 computer. In the present work with the 80 elements, 1545 degrees of freedom mesh a total of 1530 CPU seconds were required for final convergence. Almost one third of this ie. 500 seconds are needed for computation of $[G]_m$ matrices for all elements on the boundary of the mesh.

CHAPTER V

5.1 Conclusions

Need of an efficient algorithm to solve problems of practical importance is evident from the discussion in Chapter I. In the present work stress intensity factors at the crack front of an elliptical embedded flaw, in a finite thickness plate was solved using Finite Element Alternating Technique. Based on this work following conclusions could be drawn.

1. A medium mesh is sufficient for convergence to the true answer and a fine mesh is not required at all.
2. The technique is about ten times faster than Hybrid Finite Element Method.

5.2 Scope for Further Work

Following further work is suggested in order that more complicated component and crack geometries could be handled.

1. Use of multiple polynomial regression in step (2) of FEAT.
2. Programming equations relating to surface crack and multimode applications.
3. Coupling of the Trefftz's solution code developed, to a general FEM code e.g. SAP IV, or SESAM.

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